On the maximization of a class of functionals on convex regions, and the characterization of the farthest convex set

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Abstract

This article considers a family of functionals J to be maximized over the planar convex sets K for which the perimeter and Steiner point have been fixed. Assuming that J is the integral of a quadratic expression in the support function h, the maximizer is always either a triangle or a line segment (which can be considered as a collapsed triangle). Among the concrete consequences of the main theorem is the fact that, given any convex body K_1 of finite perimeter, the set in this class that is farthest away in the sense of the L^2 distance is always a line segment. The same property is proved for the Hausdorff distance.

Keywords: isoperimetric problem, shape optimization, convex geometry, polygons, farthest convex set

AMS classification: 52A10, 52A40, 52B60, 49Q10

1 Introduction

Given a convex set K_1 in the plane, consider the problem of finding a second convex set that is as far as possible from K_1 in the sense of usual distances like the Hausdorff distance or the L^2 distance, subject to two natural geometric constraints, *viz.*, that the two sets have the same perimeter and Steiner point, without either of which conditions there are sets arbitrarily far away from K_1. A plausible conjecture, which we prove below, is that the farthest convex set, subject to the two constraints, is always a "needle," to use the colorful terminology of Pólya and Szegő [12] for a line segment in the plane.

In the case of the L^2 distance, the problem of the farthest convex set can be expressed as the maximization of a quadratic integral functional of the support function of the desired set, and, as we shall show, with the same two geometric constraints it is possible to characterize the maximizers of a wider class of such functionals as either triangles or needles, which, intuitively, can be considered as collapsed triangles. One of our inspirations for pursuing the wider class of functionals, the maximizers of which are triangles, is a recent article [8], in which the maximizers of another class of convex functionals were shown to be polygons. Now, the maximizers of a convex functional must lie on the boundary of the feasible set, which is to say, in our case or that of [8], that the maximizers will be non strictly convex, but not simple polygons *a priori*. What restrictions are needed on the functional to imply furthermore that the maximizer must be triangular? In this article, we consider functionals that are expressible as integrals of quadratic expressions in the support function, and show that the maximizers are always generalized triangles, i.e., triangles or needles.

An advantage of describing shape-optimization problems through the support function h is that it is easy to express many geometric features, including perimeter and area, in terms of h. Yet another tool that is available in the case of functionals that are quadratic in h is that of Fourier series [3], because through the Parseval relation it is possible to rewrite many such functionals as series with geometric properties accessible through the form of the coefficients. Indeed another one of our inspirations was the analysis of the maximizers of the L^2 means of chord lengths of curves through Fourier series found in [2, 1]. When the means with respect to arc length are replaced with means weighted by curvature, the problem falls within the category of quadratic functionals of h considered in this article. Interestingly, the cases of optimality of the weighted and unweighted problems are completely different. Because additional analysis is possible for quadratic functionals when the coefficients in the equivalent series enjoy certain properties, we shall defer details on the chord problem to a future article [5].

This paper is organized as follows. We begin Section 2 with the main notation and general optimality conditions. We state our main result in

Subsection 2.3. Next, Section 3 is devoted to the problem of finding the farthest convex set. We begin with an inequality involving the minimum and the maximum of the support function, in the spirit of [10]. We then consider the case of the Hausdorff distance, and we finish with the case of the L^2 distance, for which our main result is essential.

2 Notation and preliminary results

2.1 Notation

When convenient $\mathbb{R}^2$ will be identified with the complex plane, and the dot product of two vectors $\mathbf{x}$ and $\mathbf{w}$ with $\Re(x\,\overline{w})$. Let $\mathbb{T}$ be the unit circle, identified with $[0, 2\pi)$. For $\theta \in \mathbb{T}$, we shall denote by $h_K(\theta)$ (or more simply $h(\theta)$ if not ambiguous) the support function of the convex set K; we recall that by definition $h(\theta)$ is the distance from the origin to the support line of K having outward unit normal $e^{i\theta}$:

$$h_K(\theta) := \max\{x \cdot e^{i\theta} : x \in K\}\,.$$

It is known that the boundary of a planar convex set has at most a countable number of points of nondifferentiability. More precisely, the two directional derivatives of the the function defining any portion of the boundary exist at every point and their difference is uniformly bounded. We refer to [13, 15] for this and other standard facts about convex regions. It follows from the regularity of the boundary that the support function h belongs to the periodic Sobolev space $H^1(\mathbb{T})$.

For a polygon K with n sides, we let $a_1, a_2, \ldots, a_n$ and $\theta_1, \theta_2, \ldots, \theta_n$ denote the lengths of the sides and the angles of the corresponding outer normals. The following characterization of the support function of such a polygon is classical and will be useful here:

Proposition 2.1. *With the notation given above, the support function of the polygon K satisfies*

$$\frac{d^2 h_K}{d\theta^2} + h_K = \sum_{j=1}^{n} a_j \delta_{\theta_j}\,, \tag{1}$$

where the derivative is to be understood in the sense of distributions, and δ_{θ_j} stands for a Dirac measure at point θ_j.

Eq. (1) can be proved by a direct calculation. It is a special case of a formula of Weingarten, whereby for any support function h_K of a convex set K, $\frac{d^2 h_K}{d\theta^2} + h_K = h_K'' + h_K$ is a nonnegative measure, which is interpreted as the (generalized) radius of curvature R at the point of contact with the support line corresponding to θ. We shall denote by S_h (or S_K if we want to emphasize the dependence on the convex set K) the support of this measure. It will be useful to recover the support function from the radius of curvature. This can be accomplished by solving the ordinary differential equation:

$$h'' + h = R \tag{2}$$

for a 2π-periodic function $h(\theta)$ subject to the conditions

$$\int_0^{2\pi} h(\theta) \cos\theta \, d\theta = \int_0^{2\pi} h(\theta) \sin\theta \, d\theta = 0\,. \tag{3}$$

These orthogonality conditions are imposed because (2) is in the second Fredholm alternative and hence needs such conditions for uniqueness. They can always be arranged by a choice of the origin, *viz.*, that it is fixed at the Steiner point $s(K)$. Recall that the Steiner point $s(K)$ of a convex planar set K is defined by

$$s(K) = \frac{1}{\pi} \int_0^{2\pi} h_K(\theta) e^{i\theta} \, d\theta\,. \tag{4}$$

By Fredholm's condition for existence the function or measure $R(\theta)$ on the right side of (2) must satisfy the same orthogonality, that is,

$$\int_0^{2\pi} R \cos\theta \, d\theta = \int_0^{2\pi} R \sin\theta \, d\theta = 0.$$

Since these restrictions on the radius of curvature are necessary conditions in any case for the closure of the boundary curve of K, they are automatically fulfilled.

An explicit Green function can be found to solve (2) for h if R is given, *viz.*, with $G(t) := \frac{1}{2}\left(1 - \frac{|t|}{\pi}\right) \sin|t|$,

$$h(\theta) = \frac{1}{2} \int_{-\pi}^{\pi} G(t) R(\theta + t) \, dt\,. \tag{5}$$

The perimeter $P(K)$ of the convex set can be easily calculated from h_K:

$$P(K) = \int_0^{2\pi} h_K(\theta) \, d\theta\,. \tag{6}$$

In this article, we work within the class of convex sets whose Steiner point is at the origin and whose perimeter $P(K)$ is fixed, at a value that can be chosen as 2π without loss of generality:

$$\mathcal{A} := \{K \text{ convex set in } \mathbb{R}^2, s(K) = O, P(K) = 2\pi\}. \tag{7}$$

Given that convexity is equivalent to the nonnegativity of the radius of curvature $R = h'' + h$ (in the sense of measures), the geometric set $\mathcal{A}$ can be described in analytic terms by requiring h to lie in the function space:

$$\begin{aligned}\mathcal{H} := \{&h \in H^1(\mathbb{T}), h \geq 0,\ h'' + h \geq 0, \\ &\textstyle\int_0^{2\pi} h d\theta = 2\pi,\ \int_0^{2\pi} h \cos\theta d\theta = \int_0^{2\pi} h \sin\theta d\theta = 0\}.\end{aligned} \tag{8}$$

The class $\mathcal{A}$ contains in particular “needles,” i.e., line segments, which we regard as degenerate convex bodies in the sense that the perimeter of the segment is taken as twice its length. We shall let Σ_α designate the segment $[-i\frac{\pi}{2}e^{i\alpha}, i\frac{\pi}{2}e^{i\alpha}]$. Its support function is given by

$$h_\alpha(\theta) := \frac{\pi}{2}|\sin(\theta-\alpha)|\,, \tag{9}$$

which satisfies ${h_\alpha}'' + h_\alpha = \pi(\delta_\alpha + \delta_{\pi+\alpha})$.

2.2 Optimality conditions

If the goal is to maximize a functional J defined on the geometric class $\mathcal{A}$, and J is expressible in terms of the support function h, then we may equivalently consider the problem of determining

$$\max\{J(h) : h \in \mathcal{H}\}. \tag{10}$$

We may then analytically derive the conditions for optimality of J.

The Steiner point s of a closed convex set always lies within the set, and in the case of a convex body (a convex set of nonempty interior), s is an interior point; see, e.g., (1.7.6) in [14]. It follows that the support function of K can vanish only if K is a segment. For any convex body in $\mathcal{A}$, $h_K(\theta) > 0$ for all θ.

We next derive the first and second order optimality conditions assuming that the optimal set is not a segment, following [8]. Because the Steiner point has been fixed, the optimality conditions are with respect to variations in the subspace of functions that are L^2-orthogonal to $\{e^{i\theta}, e^{-i\theta}\}$. Alternatively, we could have imposed this constraint by introducing two additional Lagrange multipliers.

Theorem 2.2. *If $h_0 > 0$ is a solution of (10), where $J : H^1(\mathbb{T}) \to \mathbb{R}$ is C^2, then there exist $\xi_0 \in H^1(\mathbb{T})$, $\xi_0 \le 0$, and $\mu \in \mathbb{R}$ such that*

$$\xi_0 = 0 \text{ on } S_{h_0}, \tag{11}$$

and for all $v \in H^1(\mathbb{T})$ with $\int_0^{2\pi} v(\theta)e^{\pm i\theta}d\theta = 0$,

$$\langle J'(h_0), v\rangle = \langle \xi_0 + {\xi_0}'', v\rangle + \mu\int_0^{2\pi} v\,d\theta\,. \tag{12}$$

Moreover, if $v \in H^1(\mathbb{T})$ and $\lambda \in \mathbb{R}$ satisfy

$$\begin{aligned} &v'' + v \ge \lambda({h_0}'' + h_0), \\ &v \ge \lambda h_0, \\ &\langle \xi_0 + {\xi_0}'', v\rangle + \mu\textstyle\int_0^{2\pi} v\,d\theta = 0, \end{aligned} \tag{13}$$

then

$$\langle J''(h_0), v, v\rangle \le 0\,. \tag{14}$$

The proof of the foregoing theorem is classical and can be achieved using standard first and second order optimality conditions in infinite dimension space as in [11]; we refer to [8] for technical details.

Remark 1. If the optimal domain K_0 is a segment, then the optimality condition is more complicated, because the constraint $h \geq 0$ needs to be taken into account. Since it will not be needed here, we do not write the explicit form.

2.3 Integral functionals

In this section we are interested in quadratic functionals involving the support function and its first derivative. Let J be the functional defined by

$$J(K) := \int_0^{2\pi} \left(a\, h_K^2 + b\, {h'_K}^2 + c\, h_K + d\, h'_K \right) d\theta, \tag{15}$$

where a and b are nonnegative bounded functions of θ, one of them being positive almost everywhere on $\mathbb{T}$. The functions c, d are assumed to be bounded. Our main theorem is the following:

Theorem 2.3. *Every local maximizer within the class $\mathcal{A}$ of the functional J defined in* (15) *is either a line segment or a triangle.*

Proof. Let K be a local maximizer of the functional J. We have to prove that the support S_K of the measure $h''_K + h_K$ contains no more than three points. We follow ideas contained in [7] and [8].

Assume, for the purpose of a contradiction, that S_K contains at least four points $\theta_1 < \theta_2 < \theta_3 < \theta_4$ in $(0, 2\pi)$. Choose $\varepsilon > 0$ sufficiently small and such that $\theta_4 + \varepsilon - (\theta_1 - \varepsilon) < 2\pi$, and let ρ_i be any positive Borel measures that are absolutely continuous with respect to $h''_K + h_K$ and supported in nonoverlapping intervals $(\theta_i - \frac{\epsilon}{2}, \theta_i + \frac{\epsilon}{2})$. (For example, if $h''_K + h_K$ contains a point mass at θ_i, we may choose $\rho_i = \delta_{\theta_i}$, the Dirac measure at point θ_i.) We solve the four differential equations

$$\begin{cases} v''_i + v_i = \rho_i & \theta \in (\theta_1 - \varepsilon, \theta_4 + \varepsilon) \\ v_i(\theta_1 - \varepsilon) = v_i(\theta_4 + \varepsilon) = 0, \end{cases} \tag{16}$$

Note that equations (16) have unique solutions since we avoid the first eigenvalue of the interval. We also extend each function v_i by 0 outside $(\theta_1 - \varepsilon, \theta_4 + \varepsilon)$. We can always find four numbers λ_i, $i = 1, \ldots, 4$ such that the three following conditions hold, where we denote by w the function defined by $w = \sum_{i=1}^4 \lambda_i v_i$:

$$w'(\theta_1 - \varepsilon) = w'(\theta_4 + \varepsilon) = 0, \qquad \int_0^{2\pi} w\, d\theta = 0\,.$$

Then the function w solves $w'' + w = \sum_{i=1}^4 \lambda_i \rho_i$ globally on $(0, 2\pi)$. Furthermore, since $e^{\pm i\theta}$ solve the associated homogeneous equation, we can subtract multiples of these functions from w to find another such global solution, denoted v, which is orthogonal to $e^{\pm i\theta}$ and thus in the class $\mathcal{H}$. We now use the optimality conditions (11), (12) for the function v, to obtain

$$< \xi_0 + \xi_0'', v > + \mu \int_0^{2\pi} v \, d\theta = < v'' + v, \xi_0 > = \sum_{i=1}^4 \lambda_i \xi_i = 0,$$

where $\xi_i := \int_0^{2\pi} \xi_0(\theta) d\rho_i$. Therefore, v is admissible for the second order optimality condition (it is immediate to check that the two first conditions in (13) are satisfied by choosing $\lambda < 0$ with $|\lambda|$ large enough). Since the functional J is quadratic, however, this would imply that $\int_0^{2\pi} (av'^2 + bv^2) \, d\theta \leq 0$, which is impossible by the assumptions on a and b. $\square$

Remark 2. The examples given in the next section may give the impression that the maximizers for such functionals are always segments. This is not the case. Indeed, if we choose $a = c = d = 0$ and let b be a (positive) function equal to one on an ε-neighborhood of $0, 2\pi/3$ and $4\pi/3$ and very small elsewhere, the value for the equilateral triangle is of order $12\pi^2\varepsilon/27$ while the value for the best segment is of order $\pi^2\varepsilon/4$.

3 The farthest convex set

3.1 Introduction

There are many ways to define the distance between convex sets. The most familiar of these is the Hausdorff distance:

$$d_H(K, L) := \max\{\rho(K, L), \rho(L, K)\},$$

where ρ is defined by

$$\rho(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|.$$

(For a survey of possible metrics we refer to [4]; for a detailed study of the Hausdorff distance see [6]). It is remarkable that the Hausdorff distance can also be defined using the support functions, as $d_H(K, L) = \|h_K - h_L\|_\infty$. Moreover the support function allows a definition of L^2 distance, introduced by McClure and Vitale in [9], by

$$d_2(K, L) := \left(\int_0^{2\pi} |h_K - h_L|^2 \, d\theta \right)^{1/2}.$$

In [10], P. McMullen was able to determine the *diameter in the sense of the Hausdorff distance* of the class $\mathcal{A}$ in any dimension. Specifically, he proved that all sets in $\mathcal{A}$ are contained in the ball of radius $\pi/2$ centered at the origin. In terms of the support function, this means that, for any convex set K in $\mathcal{A}$, the maximum of h_K is at most $\pi/2$ (*i.e.* $P(K)/4$). We shall need the following more precise result:

Theorem 3.1. *Let K be any planar convex set the Steiner point of which is fixed at the origin. Then*

$$\max h_K \leq \frac{P(K)}{4} \leq \min h_K + \max h_K, \tag{17}$$

where both inequalities are sharp: Equality is attained by any line segment.

Proof. The first inequality in (17) is due to McMullen, who proved it in any dimension; see Theorem 1 in [10]. Let us prove the second inequality. Letting B denote the unit ball, we introduce

$$\max h_K = \tau(K) := \min\{\tau > 0 : K \subset \tau B\},$$

$$\min h_K = \rho(K) := \max\{\rho > 0 : \rho B \subset K\}.$$

The function $\tau(K)$ is convex with respect to the Minkowski sum, which can be defined with the support function via

$$h_{aK+bL} = ah_K + bh_L.$$

In contrast, the function $\rho(K)$ is concave, and as we are interested in the sum $F(K) := \tau(K) + \rho(K)$ we can call upon no particular convexity property. The minimum of h_K is attained at some point we call P and the maximum at some point Q (see Figure 2). Let us denote by L the line containing the points O and P and by σ_L the reflection across L. If we replace the convex set K by $\frac{1}{2}K + \frac{1}{2}\sigma_L(K)$, then we keep the Steiner point at the origin, we preserve the perimeter, and we decrease τ, because of convexity, without changing ρ. Therefore, to look for the minimum of $F(K)$, we can restrict ourselves to convex sets symmetric with respect to the line L passing through the point where h_K attains its minimum. Now, let S be the segment in the class $\mathcal{A}$ that is orthogonal to the line L.

We introduce the family of convex sets $K_t := tK + (1-t)S$ and study the behavior of $t \mapsto F(K_t)$. Since the ball $t\rho(K)B$ is contained in K_t and touches its boundary at tP, we know that $\rho(K_t) = t\rho(K)$. Moreover, by convexity $\tau(K_t) \leq t\tau(K) + (1-t)\tau(S)$. Therefore, since $\tau(S) = F(S)$,

$$F(K_t) \leq tF(K) + (1-t)F(S). \tag{18}$$

In particular, this implies that if $F(K) < F(S)$, it would follow that $F(K_t) < F(S)$ for t near 0. Thus, to prove the result it suffices to prove that a segment is a local minimizer for J. Without loss of generality, we consider the segment Σ_0 and perturbations of Σ_0 preserving the symmetry with respect to the line $\theta = 0$. Let us therefore consider a perturbation of the segment Σ_0, replacing its "radius of curvature" $R_0 = \pi(\delta_0 + \delta_\pi)$ by

$$R_t = R_0 + t[\varphi(x) - (\beta\delta_0 + (1-\beta)\delta_\pi)],$$

where $\varphi(x)$ is a non-negative measure. Since we can work in the class of symmetric convex sets, we may assume φ to be even. Moreover, we have to assume that $\int_0^{2\pi} R_t = 2\pi$ and $\int_0^{2\pi} R_t \cos(\theta) = 0$. (The last relation $\int_0^{2\pi} R_t \sin(\theta) = 0$ is true by symmetry). This implies that

$$\begin{array}{c} \int_0^{2\pi} \varphi = 1, \quad \text{or} \quad \int_0^{\pi} \varphi = \frac{1}{2}\,, \\ \int_0^{2\pi} \varphi\cos\theta = 2\beta - 1, \quad \text{or} \quad \beta = \frac{1}{2} + \int_0^{\pi} \varphi\cos\theta\,. \end{array} \tag{19}$$

Observe that the support function h_t of the perturbed convex set can be obtained thanks to formula (5):

$$h_t(\theta) = \frac{\pi}{2}|\sin\theta| + t\left\{\int_{-\pi}^{\pi} G(\tau)\varphi(\theta+\tau)\,d\tau - \beta G(\theta) - (1-\beta)G(\theta-\pi)\right\},$$

where G denotes the Green function. The function h_t will have its maximum near $\pi/2$, so to first order,

$$\max h_t = h_t(\frac{\pi}{2}) + o(t) = \frac{\pi}{2} + t\left\{\int_{-\pi}^{\pi} G(\tau)\varphi(\tau+\frac{\pi}{2})\,d\tau - \frac{1}{2}\right\} + o(t)\,. \tag{20}$$

In the same way, the minimum of h_t will be attained near 0 or near π, so to first order,

$$\begin{aligned} \min h_t = \min(h_t(0), h_t(\pi)) + o(t) = \\ t\min\left\{\int_{-\pi}^{\pi} G(\tau)\varphi(\tau)\,d\tau, \int_{-\pi}^{\pi} G(\tau)\varphi(\tau+\pi)\,d\tau\right\} + o(t)\,. \end{aligned} \tag{21}$$

Therefore, we must prove that

$$\int_{-\pi}^{\pi} G(\tau)\varphi(\tau+\frac{\pi}{2})\,d\tau + \int_{-\pi}^{\pi} G(\tau)\varphi(\tau)\,d\tau - \frac{1}{2} > 0 \tag{22}$$

and

$$\int_{-\pi}^{\pi} G(\tau)\varphi(\tau+\frac{\pi}{2})\,d\tau + \int_{-\pi}^{\pi} G(\tau)\varphi(\tau+\pi)\,d\tau - \frac{1}{2} > 0\,. \tag{23}$$

Let us prove for example (22); the other inequality is similar. Letting

$$A := \int_{-\pi}^{\pi} G(\tau)\varphi(\tau+\frac{\pi}{2})\,d\tau + \int_{-\pi}^{\pi} G(\tau)\varphi(\tau)\,d\tau = \int_{-\pi}^{\pi} (G(\tau)+G(\tau-\frac{\pi}{2}))\varphi(\tau)\,d\tau$$

and using the fact that φ is even,

$$A = \int_0^{\pi} [G(\tau) + G(\tau-\frac{\pi}{2}) + G(-\tau) + G(-\tau-\frac{\pi}{2})]\varphi(\tau)\,d\tau\,.$$

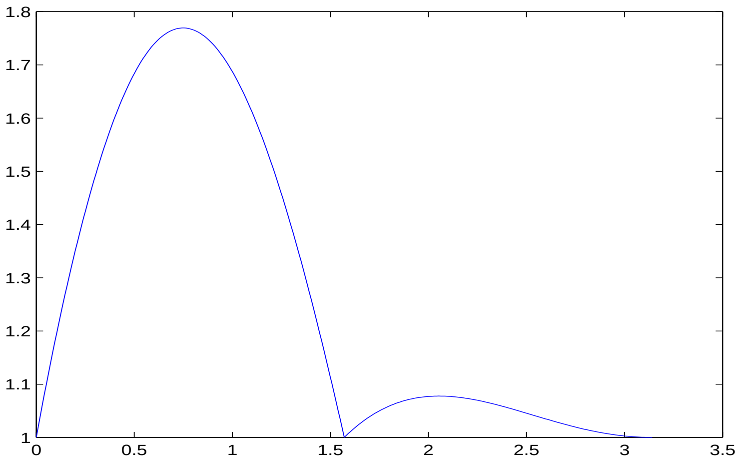

Figure 1: The function $\tau \mapsto G(\tau) + G(\tau - \frac{\pi}{2}) + G(-\tau) + G(-\tau - \frac{\pi}{2})$.

It is elementary to check that the function $\tau \mapsto G_4(\tau) := G(\tau) + G(\tau - \frac{\pi}{2}) + G(-\tau) + G(-\tau - \frac{\pi}{2})$ is always greater than or equal to one (see Figure 1), so $A \geq \int_0^\pi \varphi(\tau)\, d\tau = \frac{1}{2}$. Moreover, since the function G_4 is equal to one only for $\tau = 0, \pi/2$ or π, the inequality will be strict unless the support of φ is concentrated at the four points $-\pi/2, 0, \pi/2, \pi$. This last case actually corresponds to a (thin) rectangle $K_\alpha = [-\alpha, \alpha] \times [-\pi/2 + \alpha, \pi/2 - \alpha]$, for which a direct computation shows that $\min h_{K_\alpha} = \alpha/2$ and $\max h_{K_\alpha} = \left(\alpha^2 + (\pi - \alpha)^2\right)^{1/2}/2$, and $F(K_\alpha) > \pi/2 = F(S)$ follows immediately. $\square$

Another consequence of McMullen's result cited above is that the Hausdorff distance between two sets in $\mathcal{A}$ is always less or equal to $\pi/2$, the upper bound being obtained by two orthogonal segments.

In the remainder of this section, we address the question of finding the *farthest convex set* in the class $\mathcal{A}$ from a given convex set, as measured by either of the two distances defined above. More exactly, letting C be a given convex set in the class $\mathcal{A}$, we wish to find the convex set K_C such that

$$d(C, K_C) = \max\{d(C, K) : K \in \mathcal{A}\},$$

where d may stand either for d_H or for d_2.

First of all, let us give an existence result for such a problem.

Theorem 3.2. *Let $d(.,.)$ be a distance function for convex sets that behaves continuously under uniform convergence of the support functions. Then the problem*

$$\max\{d(C, K) : K \in \mathcal{A}\} \tag{24}$$

has a solution.

Proof. For the proof we will use the following lemma:

Lemma 3.3. *For any h in the set $\mathcal{H}$ (defined in (8)),*

$$\|h\|_{H^1}^2 := \int_0^{2\pi} \left(|h|^2 + |h'|^2\right) d\theta \le 16\pi/3 .$$

Proof of the lemma. For any h in $\mathcal{H}$, we have

$$0 \le \int_0^{2\pi} h(h + h'')\, d\theta = \int_0^{2\pi} h^2\, d\theta - \int_0^{2\pi} h'^2\, d\theta . \tag{25}$$

We now use the fact that the first eigenvalues of the problem

$$\begin{cases} -h'' = \lambda h \\ h \ 2\pi\text{-periodic} \end{cases}$$

are 0 (associated with the constant eigenfunction), 1 (of multiplicity 2 associated with $\sin\theta$ and $\cos\theta$), and 4 (of multiplicity 2 associated with $\sin 2\theta$ and $\cos 2\theta$). Thus, on $\mathcal{A}$ we can write a minimizing formula:

$$4 = \min_{v \in \mathcal{A}} \left\{ \frac{\int_0^{2\pi} v'^2\, d\theta}{\int_0^{2\pi} v^2\, d\theta} \ s.t. \int_0^{2\pi} v = \int_0^{2\pi} v\cos\theta = \int_0^{2\pi} v \sin\theta = 0 \right\} . \tag{26}$$

Applying (26) to $v = h - 1$ yields

$$\int_0^{2\pi} h'^2 \ge 4 \int_0^{2\pi} (h-1)^2 = 4 \int_0^{2\pi} h^2 - 8\pi ,$$

or

$$\int_0^{2\pi} h^2 \le \frac{1}{4} \int_0^{2\pi} h'^2 + 2\pi . \tag{27}$$

Combining (25) with (27) leads to

$$\frac{3}{4} \int_0^{2\pi} h^2 \le 2\pi,$$

and the result follows, once again applying (25) and summing the two last inequalities. □

We return to the proof of Theorem 3.2. Let K_n be a maximizing sequence of convex sets and h_n be the corresponding support functions. Since the perimeter of K_n is uniformly bounded and the sets K_n contain the origin, the Blaschke selection theorem applies: there exists a subsequence, still denoted with the same index, which converges in the Hausdorff sense to a convex set K. According to Lemma 3.3, the support functions h_n are bounded in $H^1(\mathbb{T})$, and consequently we may assume that the sequence converges uniformly to a function h, which is necessarily the support function of K. Finally, since the distance d has been assumed continuous for this kind of convergence, the existence of a maximizer follows. □

3.2 The farthest convex set in the Hausdorff distance

For the Hausdorff distance, we are able to prove that the farthest convex set is always a segment:

Theorem 3.4. *If C is a given convex set in the class $\mathcal{A}$, then the convex set K_C for which*

$$d_H(C, K_C) = \max\{d_H(C, K) : K \in \mathcal{A}\}$$

is a segment. More precisely, it is a segment orthogonal to any line OQ, where Q is a point at which h_C is maximal.

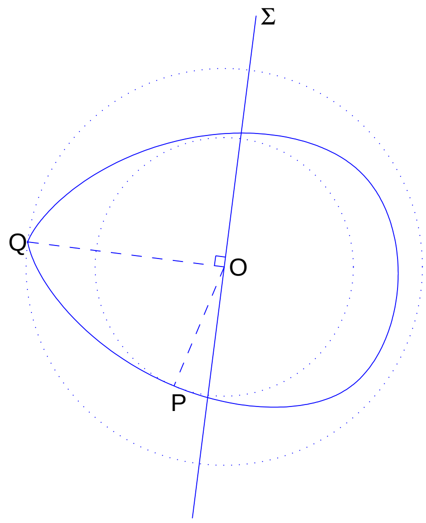

Figure 2: The farthest segment Σ for the Hausdorff distance.

Proof. Let B_1 be the largest ball centered at O and contained in C and B_2 the smallest ball centered at O that contains C. We denote by R_1 (resp. R_2) the radius of B_1 (resp. B_2). Let P, resp. Q, be contact points of these balls with the boundary of C (see Figure 2). We also denote by Σ_1 the segment (centered at 0) containing P and by Σ the segment (centered at 0) orthogonal to OQ.

It is easy to see that Σ_1 is optimal, among all segments S, to maximize $\rho(S, C)$ while Σ is optimal to maximize $\rho(C, S)$. Next we shall prove that, for any convex set K in $\mathcal{A}$:

$$\rho(K, C) \leq \rho(\Sigma_1, C) \quad \text{and} \quad \rho(C, K) \leq \rho(C, \Sigma)\,. \tag{28}$$

For the first inequality, let us consider any point M in K. By construction of the ball B_1:

$$d(M,C) \leq d(M,B_1) = OM - R_1 \,.$$

Since by the first inequality of Theorem 3.1, $OM \leq P(K)/4 = \pi/2$, the result follows taking the supremum in M since $\rho(\Sigma_1, C) = \pi/2 - R_1$.

We prove now the second inequality in (28) for any convex body K (the result is already clear for segments as mentioned above). Since the Steiner point lies in the interior, for any point $M \in \partial C$,

$$d(M,K) < OM \leq OQ = \rho(C,\Sigma) \,.$$

Therefore, taking the supremum in M, $\rho(C,K) \leq \rho(C,\Sigma)$.
From (28) it follows that for any set K:

$$d_H(K,C) \leq \max(d_H(\Sigma_1,C), d_H(\Sigma,C)) \,.$$

Now, we use the second inequality in Theorem 3.1, which can be written

$$\rho(\Sigma_1,C) = \pi/2 - R_1 \leq R_2 = \rho(C,\Sigma) \,.$$

Since, however, $\rho(C,\Sigma_1) \leq \rho(C,\Sigma)$, we have

$$d_H(\Sigma_1,C) \leq \rho(C,\Sigma) \leq d_H(\Sigma,C),$$

which yields the desired result. □

Remark 3. As suggested by the referee, it may be interesting to extend the previous results to any dimension. Following McMullen (see [10]), the good class of convex sets seems to be

$$\mathcal{A}_N := \{K \text{ convex set in } \mathbb{R}^N, s(K) = O, w(K) = 2\} \tag{29}$$

where $s(K)$ is the Steiner point and $w(K)$ is the *intrinsic width*, defined as

$$w(K) = \frac{1}{\kappa_{N-1}} \int_{S^{N-1}} h_K \, d\sigma$$

where κ_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$ and S^{N-1} is the unit sphere in $\mathbb{R}^N$. In particular, the segments in the class $\mathcal{A}_N$ are centered at 0 and have length 2.

As in the two-dimensional case, we have

Theorem 3.5. *For any convex set C in the class $\mathcal{A}_N$, the convex set K_C that realizes*

$$d_H(C,K_C) = \max\{d_H(C,K) : K \in \mathcal{A}_N\}$$

is a segment.

The difference from Theorem 3.4 is that in general we cannot characterize the segment that achieves the maximum. The proof begins as before: Keeping the same notation, we can prove the two inequalities, for any $K \in \mathcal{A}_N$:

$$\rho(K, C) \leq \rho(\Sigma_1, C) \quad \text{and} \quad \rho(C, K) \leq \rho(C, \Sigma)\,.$$

It is clear that the quantity $d_H(S, C)$ has a maximizer among all segments S in the class $\mathcal{A}_N$, say S_0, and that

$$d_H(K, C) \leq \max(\rho(\Sigma_1, C), \rho(C, \Sigma)) \leq \max(d_H(\Sigma_1, C), d_H(C, \Sigma)) \leq d_H(S_0, C),$$

which shows that S_0 is the farthest convex set for C.

We cannot be more precise because the inequality

$$\frac{1}{2}w(K) \leq \min h_K + \max h_K \tag{30}$$

that was proved in two dimensions is no longer valid in higher dimension! For example, let us consider in $\mathbb{R}^3$ a (vertical) cylinder with height 2 and radius of the base $\ell \leq 1$. In spherical coordinates, its support function depends only on the angle φ and is given by $h_K(\theta, \varphi) = \ell \cos\varphi + \sin\varphi, \varphi \in [0, \pi/2]$ (by symmetry, it suffices to know it for $\varphi \in [0, \pi/2]$). In particular, we have

$$\min h_K + \max h_K = \ell + \sqrt{1 + \ell^2}$$

while

$$\frac{1}{2}w(K) = \frac{1}{2\pi}\int_{S2} h_K = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} h_K(\theta, \varphi)\cos\varphi\, d\varphi = 1 + \frac{\ell\pi}{2}$$

and it is clear that the inequality cannot hold, for any $\ell < 1$.

Nevertheless, we are able to prove a weaker inequality, namely

$$\frac{1}{2}w(K) \leq \frac{\pi}{2}(\min h_K + \max h_K)\,. \tag{31}$$

Indeed, let P be some point in K where h_K attains its minimum. Let Π be any plane containing the line OP and σ_Π be the reflection with respect to Π. Then, the convex set $\frac{1}{2}\left(K + \sigma_\Pi(K)\right)$ is still in the class $\mathcal{A}_3$, its support function has the same minimum and a smaller maximum (by convexity of the maximum). Since this is true for any such plane Π, we can restrict ourselves to three dimensional convex sets which are axisymmetric with the axis OP. Now, the support function of the three dimensional axisymmetric convex set K coincides with the support function of a plane section K_0, so we can apply Theorem 3.1, yielding

$$\frac{1}{4}\int_{-\pi}^{\pi} h_{K_0} \leq \min h_{K_0} + \max h_{K_0} = \min h_K + \max h_K\,. \tag{32}$$

Since h_K only depends on the polar angle φ, we have

$$\frac{1}{2}w(K) = \frac{1}{2\pi}\int_{S^2} h_K = \int_{-\pi/2}^{\pi/2} h_{K_0}\cos\varphi\,d\varphi\,.$$

Now, the maximization problem

$$\max\{\frac{\int_{-\pi/2}^{\pi/2} h(\varphi)\cos\varphi\,d\varphi}{\int_{-\pi/2}^{\pi/2} h(\varphi)\,d\varphi},\ h+h''\geq 0,\int_{-\pi/2}^{\pi/2} h(\varphi)\sin\varphi\,d\varphi = 0\}$$

can be solved using Theorem 2.2, the solution being $h(\varphi) = \cos\varphi$. This shows that the inequality

$$\int_{-\pi/2}^{\pi/2} h_{K_0}(\varphi)\cos\varphi\,d\varphi \ \leq \frac{\pi}{4}\int_{-\pi/2}^{\pi/2} h_{K_0}(\varphi)\,d\varphi = \frac{\pi}{8}\int_{-\pi}^{\pi} h_{K_0}(\varphi)\,d\varphi \qquad (33)$$

holds, and the result follows from (32) and (33).

More generally, in any dimension, one can state an inequality of the form

$$\frac{1}{2}w(K) \leq c_N^*(\min h_K + \max h_K),$$

and it would be interesting to know the optimal constant c_N^* in dimension N. We have shown that $c_2^* = 1$ and that $\frac{\pi}{4}+\frac{1}{\pi} \leq c_3^* \leq \frac{\pi}{2}$. (The first inequality comes from the optimal cylinder.) It would be also very interesting to know which convex set saturates the inequality, since it is definitely not a segment in dimension $N \geq 3$.

3.3 The farthest convex set for the L^2 distance

For the L^2 distance, the result is similar: the convex set farthest from any given convex set will be a segment. The proof is more complicated and relies on Theorem 2.3.

Theorem 3.6. *For any given convex set C in the class $\mathcal{A}$, the convex set K_C for which*

$$d_2(C, K_C) = \max\{d_2(C,K) : K \in \mathcal{A}\}$$

is a segment. More precisely, it is any segment Σ_α for which α maximizes the one-variable function $\alpha \mapsto \int_0^\pi h_C(\theta+\alpha)\sin\theta\,d\theta$.

Proof. Fix a convex set C in the class $\mathcal{A}$. An immediate consequence of Theorem 2.3 applied to the functional J defined by

$$J(K) = \int_0^{2\pi}(h_K - h_C)^2\,d\theta = \int_0^{2\pi}(h_K^2 - 2h_Ch_K + h_C^2)\,d\theta$$

is that the farthest convex set is either a triangle or a segment. Thus, to prove the result we need to exclude the first possibility.

Let T be a triangle that we assume to be a critical point for the functional $J : K \mapsto d_2^2(C, K)$. Each triangle in the class $\mathcal{A}$ will be uniquely characterized by its three angles $(\theta_1, \theta_2, \theta_3)$ such that $e^{i\theta_k}$ is the normal vector to each side. The only restrictions we need to put on these angles are

$$0 < \theta_2 - \theta_1 < \pi,\ 0 < \theta_3 - \theta_2 < \pi,\ 0 < 2\pi + \theta_1 - \theta_3 < \pi\,. \tag{34}$$

The lengths of the sides will be denoted by a_1, a_2, a_3. According to the Law of Sines, given that the perimeter of T is 2π, the three lengths are given by:

$$\begin{aligned} a_1 &= \frac{2\pi \sin(\theta_3 - \theta_2)}{\sin(\theta_3 - \theta_2) + \sin(\theta_2 - \theta_1) + \sin(\theta_1 - \theta_3)}, \\ a_2 &= \frac{2\pi \sin(\theta_1 - \theta_3)}{\sin(\theta_3 - \theta_2) + \sin(\theta_2 - \theta_1) + \sin(\theta_1 - \theta_3)}, \\ a_3 &= \frac{2\pi \sin(\theta_2 - \theta_1)}{\sin(\theta_3 - \theta_2) + \sin(\theta_2 - \theta_1) + \sin(\theta_1 - \theta_3)}\,. \end{aligned} \tag{35}$$

Note that the denominator $\sin(\theta_3-\theta_2)+\sin(\theta_2-\theta_1)+\sin(\theta_1-\theta_3)$ can also be written $4\sin((\theta_3-\theta_2)/2)\sin((\theta_2-\theta_1)/2)\sin((\theta_1-\theta_3)/2)$.

If A_1, A_2, A_3 denote the vertices of the triangle, then from the relation $\vec{A_1A_2} + \vec{A_2A_3} + \vec{A_3A_1} = \vec{0}$ rotated by $\pi/2$, we get

$$a_1\cos\theta_1 + a_2\cos\theta_2 + a_3\cos\theta_3 = 0 \quad \text{and} \quad a_1\sin\theta_1 + a_2\sin\theta_2 + a_3\sin\theta_3 = 0\,. \tag{36}$$

With the Steiner point at the origin, the support function $h_T(\theta)$ of the triangle T can be calculated with the aid of formula (5), using the fact that the radius of curvature of T is given by $R = a_1\delta_{\theta_1} + a_2\delta_{\theta_2} + a_3\delta_{\theta_3}$, according to (1). One possible expression for h is:

$$h_T(\theta) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\sum_{k=1}^{3} a_k\theta_k\sin(\theta-\theta_k), & \theta \le \theta_1 \text{ or } \theta \ge \theta_3 \\ \dfrac{1}{2\pi}\displaystyle\sum_{k=1}^{3} a_k\theta_k\sin(\theta-\theta_k) + a_1\sin(\theta-\theta_1), & \theta_1 \le \theta \le \theta_2 \\ \dfrac{1}{2\pi}\displaystyle\sum_{k=1}^{3} a_k\theta_k\sin(\theta-\theta_k) - a_3\sin(\theta-\theta_3), & \theta_2 \le \theta \le \theta_3, \end{cases} \tag{37}$$

where we have used the fact that, by (36), for any θ, $\sum_{k=1}^{3} a_k\sin(\theta-\theta_k) = 0$. We denote by $\phi(\theta)$ the function

$$\phi(\theta) = \frac{1}{2\pi}\sum_{k=1}^{3} a_k\theta_k\sin(\theta-\theta_k)\,.$$

Now, if T is a critical point of the functional $\int_0^{2\pi}(h_K - h_C)^2\, d\theta$ among any convex set in $\mathcal{A}$, it is also a critical point among triangles. So we can express that the derivatives with respect to $\theta_1, \theta_2, \theta_3$ of

$$J(\theta_1, \theta_2, \theta_3) = \int_0^{2\pi} (h_T - h_C)^2\, d\theta\,,$$

where h_T is defined in (37), are zero, that is

$$\int_0^{2\pi} (h_T - h_C) \frac{\partial h_T}{\partial \theta_j}\, d\theta = 0,\ j = 1, 2, 3\,.$$

According to (37), we have (note that h_T is continuous):

$$\begin{aligned}
\frac{\partial h_T}{\partial \theta_1} &= \frac{\partial \phi}{\partial \theta_1} + (\frac{\partial a_1}{\partial \theta_1} \sin(\theta - \theta_1) - a_1 \cos(\theta - \theta_1))\chi_{[\theta_1,\theta_2]} - \frac{\partial a_3}{\partial \theta_1} \sin(\theta - \theta_3)\chi_{[\theta_2,\theta_3]}\,,\\
\frac{\partial h_T}{\partial \theta_2} &= \frac{\partial \phi}{\partial \theta_2} + \frac{\partial a_1}{\partial \theta_2} \sin(\theta - \theta_1)\chi_{[\theta_1,\theta_2]} - \frac{\partial a_3}{\partial \theta_2} \sin(\theta - \theta_3)\chi_{[\theta_2,\theta_3]}\,,\\
\frac{\partial h_T}{\partial \theta_3} &= \frac{\partial \phi}{\partial \theta_3} + \frac{\partial a_1}{\partial \theta_3} \sin(\theta - \theta_1)\chi_{[\theta_1,\theta_2]} - (\frac{\partial a_3}{\partial \theta_3} \sin(\theta - \theta_3) - a_3 \cos(\theta - \theta_3))\chi_{[\theta_2,\theta_3]}\,.
\end{aligned} \tag{38}$$

But since for $j = 1, 2, 3$, $\frac{\partial \phi}{\partial \theta_j}$ is a linear combination of $\sin(\theta - \theta_k)$ and $\cos(\theta - \theta_k)$, the contributions $\int_0^{2\pi}(h_T - h_C)\dfrac{\partial \phi}{\partial \theta_k}\, d\theta$ are zero because $\int_0^{2\pi} h \cos\theta d\theta = \int_0^{2\pi} h \sin\theta d\theta = 0$ for both h_T and h_C. Therefore, the optimality conditions at the critical triangle T can be written

$$\left\{\begin{aligned}
&\frac{\partial a_1}{\partial \theta_1} \int_{\theta_1}^{\theta_2}(h_T - h_C)\sin(\theta - \theta_1) - a_1 \int_{\theta_1}^{\theta_2}(h_T - h_C)\cos(\theta - \theta_1) -\\
&\qquad \frac{\partial a_3}{\partial \theta_1} \int_{\theta_2}^{\theta_3}(h_T - h_C)\sin(\theta - \theta_3) = 0\\
&\frac{\partial a_1}{\partial \theta_2} \int_{\theta_1}^{\theta_2}(h_T - h_C)\sin(\theta - \theta_1) - \frac{\partial a_3}{\partial \theta_2} \int_{\theta_2}^{\theta_3}(h_T - h_C)\sin(\theta - \theta_3) = 0\\
&\frac{\partial a_1}{\partial \theta_3} \int_{\theta_1}^{\theta_2}(h_T - h_C)\sin(\theta - \theta_1) + a_3 \int_{\theta_2}^{\theta_3}(h_T - h_C)\cos(\theta - \theta_3) -\\
&\qquad \frac{\partial a_3}{\partial \theta_3} \int_{\theta_2}^{\theta_3}(h_T - h_C)\sin(\theta - \theta_3) = 0\,.
\end{aligned}\right. \tag{39}$$

Using (35), we can explicitly compute each partial derivative $\frac{\partial a_i}{\partial \theta_j}$. For example, for a_1 they work out to be

$$\begin{aligned}
&\frac{\partial a_1}{\partial \theta_2} = \tfrac{\pi}{2} \cot\left(\tfrac{\theta_1 - \theta_3}{2}\right) / \sin^2\left(\tfrac{\theta_2 - \theta_1}{2}\right), \quad \frac{\partial a_1}{\partial \theta_3} = -\tfrac{\pi}{2} \cot\left(\tfrac{\theta_2 - \theta_1}{2}\right) / \sin^2\left(\tfrac{\theta_1 - \theta_3}{2}\right)\\
&\frac{\partial a_1}{\partial \theta_1} = -\frac{\partial a_1}{\partial \theta_2} - \frac{\partial a_1}{\partial \theta_3} = -\tfrac{\pi}{4} \frac{\sin(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_3)}{\sin^2 \frac{\theta_2 - \theta_1}{2} \sin^2 \frac{\theta_1 - \theta_3}{2}}\,.
\end{aligned} \tag{40}$$

In order to simplify the partial derivatives, we introduce the following integrals:

$$\begin{array}{rr} I_1 = \int_{\theta_1}^{\theta_2}(h_T - h_C)\sin(\theta-\theta_1) & I_2 = \int_{\theta_1}^{\theta_2}(h_T - h_C)\sin(\theta-\theta_2) \\ J_1 = \int_{\theta_2}^{\theta_3}(h_T - h_C)\sin(\theta-\theta_2) & J_2 = \int_{\theta_2}^{\theta_3}(h_T - h_C)\sin(\theta-\theta_3) \\ K_1 = \int_{\theta_3}^{\theta_1+2\pi}(h_T - h_C)\sin(\theta-\theta_3) & K_2 = \int_{\theta_3}^{\theta_1+2\pi}(h_T - h_C)\sin(\theta-\theta_1) \end{array} \tag{41}$$

In consequence, the second equality in (39) simplifies to:

$$\frac{1}{\sin^2\frac{\theta_2-\theta_1}{2}}\,I_1 + \frac{1}{\sin^2\frac{\theta_3-\theta_2}{2}}\,J_2 = 0 \tag{42}$$

We also introduce the integral

$$I = \int_0^{2\pi}(h_T - h_C)h_T\,d\theta, \tag{43}$$

which is nothing else than half the derivative of the functional J at h_T. Using the notation (41) and formulae (37), together with the fact that $\int_0^{2\pi}(h_T - h_C)\phi\,d\theta = 0$, we get $I = a_1 I_1 - a_3 J_2$. Thanks to (35) and (42), we can rewrite I_1 and J_2 in terms of I,

$$I = -\frac{1}{2\sin^2\frac{\theta_2-\theta_1}{2}}\,I_1 = \frac{1}{2\sin^2\frac{\theta_3-\theta_2}{2}}\,J_2\,. \tag{44}$$

Obviously, by symmetry and using other equivalent expressions of the support function h_T, we can also conclude that

$$I = -\frac{1}{2\sin^2\frac{\theta_3-\theta_2}{2}}\,J_1 = \frac{1}{2\sin^2\frac{\theta_1-\theta_3}{2}}\,K_2 = -\frac{1}{2\sin^2\frac{\theta_1-\theta_3}{2}}\,K_1 = \frac{1}{2\sin^2\frac{\theta_2-\theta_1}{2}}\,I_2\,. \tag{45}$$

Note that we can easily express any of the integrals $\int_{\theta_j}^{\theta_{j+1}}(h_T - h_C)\sin\theta\,d\theta$ or $\int_{\theta_j}^{\theta_{j+1}}(h_T - h_C)\cos\theta\,d\theta$ in terms of the six integrals defined in (41) and therefore entirely in terms of I.

Now summing the three equations in (39) and taking into account that $\frac{\partial a_1}{\partial\theta_1} + \frac{\partial a_1}{\partial\theta_2} + \frac{\partial a_1}{\partial\theta_3} = 0$, and the analogous relation for a_3, yields

$$a_3\int_{\theta_2}^{\theta_3}(h_T - h_C)\cos(\theta-\theta_3) - a_1\int_{\theta_1}^{\theta_2}(h_T - h_C)\cos(\theta-\theta_1) = 0\,.$$

We can use the previous expressions to write this last inequality in terms of the integral I, so that

$$\cos\left(\frac{\theta_3-\theta_2}{2}\right)(\sin(\theta_2-\theta_1) - \sin(\theta_1-\theta_3))\,I = 0\,. \tag{46}$$

By symmetry, we get the similar relations obtained by permutation. Since the cosine is positive (the difference between two angles is less than π), we deduce from relation (46) and its analogues that

1. either $I = 0$

2. or $\theta_3 - \theta_2 = \theta_2 - \theta_1 = 2\pi + \theta_1 - \theta_3$, that is, T is an equilateral triangle.

Now, in the case of an equilateral triangle, it is also possible to simplify the integral I. The support function h_T of the equilateral triangle $\theta_1, \theta_2 = \theta_1 + 2\pi/3, \theta_3 = \theta_1 + 4\pi/3$ is also given by:

$$h_T(\theta) = \begin{cases} \dfrac{2\pi}{3\sqrt{3}} \cos(\theta - \theta_1 - \pi/3) & \theta_1 \le \theta \le \theta_2 \\ \dfrac{2\pi}{3\sqrt{3}} \cos(\theta - \theta_1 - \pi) & \theta_2 \le \theta \le \theta_3 \\ \dfrac{2\pi}{3\sqrt{3}} \cos(\theta - \theta_1 - 5\pi/3) & \theta_3 \le \theta \le \theta_1 + 2\pi \,. \end{cases} \tag{47}$$

Then we have:

$$\begin{aligned} I = \frac{2\pi}{3\sqrt{3}} \Big(& \textstyle\int_{\theta_1}^{\theta_2} (h_T - h_C) \cos(\theta - \theta_1 - \pi/3) + \int_{\theta_2}^{\theta_3} (h_T - h_C) \cos(\theta - \theta_1 - \pi) \\ & + \textstyle\int_{\theta_3}^{\theta_1 + 2\pi} (h_T - h_C) \cos(\theta - \theta_1 - 5\pi/3) \Big) \,. \end{aligned}$$

Using the notation introduced in (41), a straightforward computation produces

$$I = \frac{2\pi}{9} \left(I_1 - I_2 + J_1 - J_2 + K_1 - K_2 \right) \,.$$

Replacing each $I_1, I_2, \ldots$ on the right side by its expression in terms of I obtained in (44), Eq. (45) yields $I = -2\pi I$. Thus, we also get $I = 0$ in this case.

To conclude the proof, it remains to show that it is impossible that $I = 0$ at a (local) maximum. Thus, let us assume that I, as defined in (43), is equal to 0. We consider the family of convex sets $K_t = (1-t)T + t\Sigma_\alpha$ where Σ_α is a segment. The derivative of $t \mapsto J(K_t, C)$ at $t = 0$ is $2\int_0^{2\pi} (h_T - h_C)(h_{\Sigma_\alpha} - h_T)\, d\theta$. Since $I = 0$, this derivative is actually

$$g(\alpha) := \pi \int_0^{2\pi} (h_T - h_C)(\theta) |\sin(\theta - \alpha)| \, d\theta \,.$$

We can also write $g(\alpha)$ as

$$g(\alpha) := 2\pi \int_0^{\pi} (h_T - h_C)(\theta + \alpha) \sin(\theta) \, d\theta \,.$$

Note that this function is π-periodic and continuous in the variable α, and that its integral over $(0, 2\pi)$ is

$$2\pi \int_0^{2\pi} \int_0^{\pi} (h_T - h_C)(\theta + \alpha) \sin(\theta) \, d\theta d\alpha = 0 \,.$$

Therefore, either $g(\alpha)$ takes both positive and negative values, in which case T cannot be a local maximizer, or else $g(\alpha)$ is identically 0. In the latter case, we come back to the optimality condition (among all convex sets) given in Theorem 2.2. There exist $\xi_0 \in H^1(\mathbb{T})$, nonpositive, vanishing on the support of T, and $\mu \in \mathbb{R}$ such that, for any $v \in H^1(\mathbb{T})$, the derivative of the functional is given by

$$< dJ(T), v >= \int_0^{2\pi} (h_T - h_C)v(\theta)\, d\theta =< \xi_0 + {\xi_0}'', v > +\mu \int_0^{2\pi} v\, d\theta\,. \quad (48)$$

Applying (48) to $v = h_{\Sigma_\alpha} - h_T$, since the left side is zero and $\int_0^{2\pi} h_{\Sigma_\alpha} = \int_0^{2\pi} h_T = 2\pi$, it follows that for any $\alpha \in (0, \pi)$, $\xi_0(\alpha) + \xi_0(\alpha + \pi) = 0$. Since $\xi_0 \leq 0$, this implies that $\xi_0 = 0$. Now applying (48) once again to $v = h_{\Sigma_\alpha}$, we get

$$0 = \int_0^{2\pi} (h_T - h_C)h_{\Sigma_\alpha}\, d\theta = 2\pi\mu.$$

Thus $\mu = 0$ and the derivative of the L^2 distance at T is identically zero. This implies that $C = T$, and is thus actually the global minimizer.

The final claim of the theorem follows easily from the expansion

$$\int_0^{2\pi} (h_{\Sigma_\alpha} - h_C)^2\, d\theta = \frac{\pi^3}{4} + \int_0^{2\pi} h_C^2\, d\theta - \pi \int_0^{2\pi} h_C |\sin(\theta - \alpha)|\, d\theta$$

and the equality

$$\int_0^{2\pi} h_C |\sin(\theta - \alpha)|\, d\theta = 2 \int_0^{\pi} h_C(\theta + \alpha) \sin\theta\, d\theta\,.$$

□

Remark 4. The farthest segment according to the L^2 distance is not necessarily unique. Apart from the trivial example of a disc, for a body of constant width, *every* segment in $\mathcal{A}$ is equally distant. This can easily be seen using the Fourier series expansion of the support function of a body C of constant width, which is known to contain only odd terms other than the constant:

$$h_C(\theta) = 1 + \sum_{k \neq -1} c_{2k+1} e^{(2k+1)i\theta},$$

while the Fourier series expansion of the support function h_α of a segment Σ_α contains only even terms. This is due to the relation ${h''}_\alpha + h_\alpha = \frac{\pi}{2}(\delta_\alpha + \delta_{\pi+\alpha})$, which when applied to $e^{-in\theta}$ yields the following equality for the n-th Fourier coefficient γ_n of h_α:

$$(1 - n^2)\gamma_n = \frac{\pi}{2} e^{-in\alpha}(1 + e^{-in\pi})\,.$$

The L^2 distance between C and Σ_α is

$$d_2(C, \Sigma_\alpha) = \int_0^{2\pi} h_\alpha^2 \, d\theta - 2 \int_0^{2\pi} h_C h_\alpha \, d\theta + \int_0^{2\pi} h_C^2 \, d\theta \,.$$

Now, using the Parseval relation and the orthogonality properties of the Fourier coefficients of the two support functions, we see that the integral $\int_0^{2\pi} h_C h_\alpha \, d\theta$ is always equal to 2π, and therefore the L^2 distance between C and a segment does not depend on the segment within the class $\mathcal{A}$.

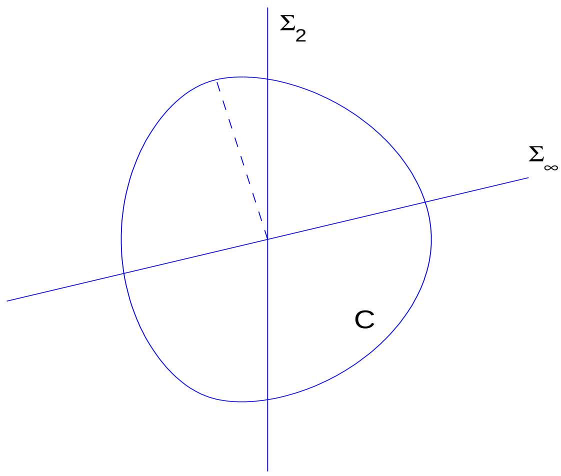

Figure 3: The farthest segments Σ_2 and Σ_∞ do not generally coincide.

Remark 5. The farthest segment for the L^2 distance and for the Hausdorff distance do not generally coincide. The Figure 3 shows the farthest segment Σ_2 (for the L^2 distance) and Σ_∞ (for the Hausdorff distance) of the convex set C whose support function is $h_C(\theta) = 1 - 0.1 \cos(2\theta) + 0.05 \cos(3\theta)$.

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